Boson representations, non-standard quantum algebras and contractions

1997 J. Phys. A: Math. Gen. 30 6797

(http://iopscience.iop.org/0305-4470/30/19/018)

J. Phys. A: Math. Gen. **30** (1997) 6797–6809. Printed in the UK PII: S0305-4470(97)79997-9

Boson representations, non-standard quantum algebras and contractions

Angel Ballesteros†, Francisco J Herranz† and Javier Negro‡

† Departamento de Física, Universidad de Burgos, Pza. Misael Bañuelos, E-09001, Burgos, Spain

‡ Departamento de Física Teórica, Universidad de Valladolid, E-47011, Valladolid, Spain

Received 29 November 1996

Abstract. A Gelfan'd–Dyson mapping is used to generate a one-boson realization for the non-standard quantum deformation of $sl(2,\mathbb{R})$ which directly provides its infinite- and finite-dimensional irreducible representations. Tensor product decompositions are worked out for some examples. Relations between contraction methods and boson realizations are also explored in several contexts. So, a class of two-boson representations for the non-standard deformation of $sl(2,\mathbb{R})$ is introduced and contracted to the non-standard quantum (1+1) Poincaré representations. Likewise, a quantum extended Hopf $sl(2,\mathbb{R})$ algebra is constructed and non-standard quantum oscillator algebra representations are obtained from it by means of another contraction procedure.

1. Introduction

Boson realizations of many symmetry algebras and superalgebras are known to be useful in many problems of condensed matter [1] and nuclear physics [2]. Among the wide variety of bosonization processes, we shall fix our attention on the so-called Gelfan'd–Dyson (GD) mapping of $sl(2,\mathbb{R})$ [3], initially introduced in spin systems. The aim of this paper is to show that deformed GD-type realizations are the most appropriate tools in order to construct the representation theory of non-standard (or h-deformed) quantum $sl(2,\mathbb{R})$, here denoted $U_z sl(2,\mathbb{R})$, and other non-standard quantum algebras linked to it by means of contraction limits. Therefore, we hope that the results reached in this paper can be directly applied to deformed shell models or coherent state methods where GD maps have been proven to be very successful.

First, we recall that the standard deformation of $sl(2,\mathbb{R})$ [4–6] is associated to the (constant) solution $r = J_+ \wedge J_-$ of the modified classical Yang–Baxter equation (YBE). This quantum algebra has been fully developed and extensively applied (see [7]). However, there also exists a non-standard deformation linked to the solution $r = J_3 \wedge J_+$ of the classical YBE. This deformation (which was first introduced at a quantum group level [8, 9], and later as a quantum Hopf algebra [10]) has recently attracted much attention. For instance, it has been applied to build up higher-dimensional non-standard quantum algebras [11] as well as the non-standard q-differential calculus [12, 13]. Its universal quantum R-matrix [14, 15] and its irreducible representations [16–18] have also been studied.

Furthermore, there exists a close relationship between $U_z sl(2,\mathbb{R})$ and the non-standard quantum $(1+1)$ Poincaré [19, 20] and oscillator algebras [21]. In particular, all of them have a similar Hopf subalgebra determined by the two generators involved in the classical r-matrix, sharing also a formally identical universal R-matrix. We will show that most of these common features can be explained by a contraction scheme connecting all these non-standard quantum algebras.

In section 2 we introduce the representation theory of $sl(2,\mathbb{R})$ with the aid of the one-boson GD realization. Afterwards, we build the one-boson infinite-dimensional representations for $U_z sl(2,\mathbb{R})$ by following the same lines. It turns out that their explicit form is somewhat more complicated than those of the standard deformation [22, 23] in the sense that they cannot be obtained by the mere substitutions of numbers by q-numbers. The corresponding finite-dimensional representations are deduced in a very natural way obtaining closed expressions for their matrix elements in any dimension. Similar results were derived in [17, 18] by using a recurrence method in another basis. In this paper, it will be shown that, as it is known for Lie algebras, boson realizations furnish us with many advantages also in the context of deformed algebras: (i) they provide us with more simple, direct and manageable expressions for the representations. (ii) They make evident how to build different classes of representations: finite, bounded or not bounded infinite-dimensional representations. (iii) They allow us to easily connect representations of non-isomorphic algebras by means of contractions. (iv) Differential (difference) realizations of the algebras under consideration are easily obtained from them.

On the other hand, these deformed representations should be fully consistent with the deformed composition of representations given by the quantum coproduct. In section 3 we illustrate such a problem for some low-dimensional representations showing that complete reducibility holds and preserves the same well known classical angular momentum decomposition rules in tensor product spaces. However, non-standard Clebsch–Gordan coefficients are shown to be essentially different to those of the standard deformation.

The remaining sections of the paper are devoted to studying the relations between deformed boson realizations and contractions. Thus, two-boson GD representations for $U_z sl(2,\mathbb{R})$ are introduced in section 4. They are shown to be the most adequate objects to obtain the representations of the non-standard quantum Poincaré algebra by means of a contraction process. In section 5 we also present a suitable quantum deformation of the (pseudo-)extended Lie algebra $sl(2,\mathbb{R})$ together with its one-boson representations so that they give rise, also through a contraction procedure, to the representations of the non-standard quantum oscillator algebra. The extension introduced here contains some interesting features that will be discussed later. Finally, some remarks close the paper.

2. One-boson $U_z sl(2,\mathbb{R})$ representations

2.1. Classical one-boson representations

To start with we shall consider the classical Lie algebra $sl(2,\mathbb{R})$

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = J_3 \tag{2.1}$$

whose irreducible representations are characterized by the eigenvalue of the quadratic Casimir element

$$\mathcal{C} = \tfrac{1}{2}J_3^2 + J_+J_- + J_-J_+. \tag{2.2}$$

If the generators $\{a_-, a_+\}$ close a boson algebra, i.e. $[a_-, a_+] = 1$, then, the realization of $sl(2, \mathbb{R})$ given by

$$J_+ = a_+ \qquad J_3 = 2a_+a_- + \beta 1 \qquad J_- = -a_+a_-^2 - \beta a_- \tag{2.3}$$

where β is a free parameter, is known as the GD one-boson realization [3, 24]. The GD map (2.3) can be used in order to easily get any of the $sl(2, \mathbb{R}) \approx su(1, 1)$ irreducible representation series [25]. For our purposes we only recall the bounded representations.

2.1.1. Lower-bounded representations. When the operators a_-, a_+ act in the usual way on the number states Hilbert space spanned by $\{|m\rangle\}_{m=0}^{\infty}$, i.e.

$$a_+|m\rangle = \sqrt{m+1}|m+1\rangle \qquad a_-|m\rangle = \sqrt{m}|m-1\rangle \tag{2.4}$$

(2.3) leads to a lower bounded representation:

$$\begin{aligned} &J_+|m\rangle = \sqrt{m+1}|m+1\rangle \qquad J_3|m\rangle = (2m+\beta)|m\rangle \\ &J_-|m\rangle = -\sqrt{m}(m-1+\beta)|m-1\rangle. \end{aligned} \tag{2.5}$$

The Casimir eigenvalue being

$$\mathcal{C} = \beta(\beta/2 - 1). \tag{2.6}$$

For negative integer values of β, hereafter denoted as $\beta_- \in \mathbb{Z}^-$, the representation (2.5) is reducible leading to a finite-dimensional irreducible quotient representation of dimension $|\beta_- - 1|$. For instance, $\beta_- = -1$ ($\mathcal{C} = \frac{3}{2}$) provides the two-dimensional representations of $sl(2, \mathbb{R})$ by setting $|2\rangle \equiv 0$:

$$\begin{aligned} &J_+|0\rangle = |1\rangle \qquad J_+|1\rangle = 0 \qquad J_3|0\rangle = -|0\rangle \\ &J_3|1\rangle = |1\rangle \qquad J_-|0\rangle = 0 \qquad J_-|1\rangle = |0\rangle. \end{aligned} \tag{2.7}$$

The numbers $\langle m|X|m'\rangle$ where $\langle m|m'\rangle = \delta_{m,m'}$ give the matrix elements of these representations; in the previous example, we have

$$J_+ = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \qquad J_- = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \qquad J_3 = \begin{pmatrix} -1 & \cdot \\ \cdot & 1 \end{pmatrix}. \tag{2.8}$$

2.1.2. Upper-bounded representations. Quite similar upper-bounded representations can be defined in the suplementary space $\{|m\rangle\}_{m=-\infty}^{-1}$. However, in order to avoid the complex numbers in the accompanying square roots (2.5) inside this space, we shall redefine the basis vectors in the form $|m\rangle \to \frac{-1}{\sqrt{m}}|m\rangle$, so that the boson operators act as

$$a_+|m\rangle = -(m+1)|m+1\rangle \qquad a_-|m\rangle = -|m-1\rangle \tag{2.9}$$

leading to the $sl(2, \mathbb{R})$ action

$$\begin{aligned} &J_+|m\rangle = -(m+1)|m+1\rangle \\ &J_3|m\rangle = (2m+\beta_+)|m\rangle \\ &J_-|m\rangle = (m-1+\beta_+)|m-1\rangle. \end{aligned} \tag{2.10}$$

The finite-dimensional representations are now originated for $\beta_+ - 2 \in \mathbb{Z}^+$, with dimension $\beta_+ - 1$. However, note that in this case the action (2.10) allows for an invariant subspace, so that it is not necessary to use the quotient mechanism to reach irreducibility.

These representations are particularly well suited for describing the differential version of the GD map (2.3),

$$J_+ = \partial_x \qquad J_3 = -2x\partial_x + \beta - 2 \qquad J_- = -x^2\partial_x + (\beta - 2)x. \tag{2.11}$$

The basis functions will be the positive integer powers $\{x^n\}_{n=0}^{+\infty}$, with the identification $x^n \equiv |{-n-1}\rangle$. In particular for the values of the label β given by $\beta_+{-2} \in \mathbb{Z}^+$ the support space for the finite $(\beta_+{-1})$-dimensional representations is generated by the monomials $\{1, x, x^2, \ldots, x^{\beta_+-2}\}$. Note also that (2.11) reproduces for $\beta = 2$ the usual differential realization of the Lie algebra of the conformal group for the one-dimensional Euclidean space.

Finite-dimensional representations obtained from the lower- (labelled by β_-) or upper-bounded ones (denoted by β_+) are equivalent whenever $\beta_+{-2} = |\beta_-| = -\beta_-$. Indeed in this case the Casimir (2.6) is the same $\mathcal{C}_{\beta_-} = \mathcal{C}_{\beta_+}$. Hereafter we shall introduce the notation $|\beta_-{-1}| = \beta_+{-1} = 2j{+1}$, where j is a half positive integer that should be identified with the label of the integer $(2j{+1})$-dimensional representations of $sl(2,\mathbb{R}) \approx su(1,1)$ [25]. Using this notation, the Casimir (2.6) will turn into the more familiar expression $\mathcal{C} = 2j(j{+1})$.

2.2. Quantum one-boson representations

The Hopf algebra $U_z sl(2,\mathbb{R})$ deforming the bialgebra generated by the classical r-matrix $r = zJ_3 \wedge J_+$ is characterized by the following coproduct, counit, antipode and commutation rules (see [15]):

$$\begin{aligned}
&\Delta(J_+) = 1 \otimes J_+ + J_+ \otimes 1 \\
&\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes \mathrm{e}^{2zJ_+} \\
&\Delta(J_-) = 1 \otimes J_- + J_- \otimes \mathrm{e}^{2zJ_+}
\end{aligned} \tag{2.12}$$

$$\epsilon(X) = 0 \qquad \text{for } X \in \{J_3, J_+, J_-\} \tag{2.13}$$

$$\gamma(J_+) = -J_+ \qquad \gamma(J_3) = -J_3\mathrm{e}^{-2zJ_+} \qquad \gamma(J_-) = -J_-\mathrm{e}^{-2zJ_+} \tag{2.14}$$

$$[J_3, J_+] = \frac{\mathrm{e}^{2zJ_+} - 1}{z} \qquad [J_3, J_-] = -2J_- + zJ_3^2 \qquad [J_+, J_-] = J_3. \tag{2.15}$$

The quantum Casimir is

$$\mathcal{C}_z = \frac{1}{2}J_3\mathrm{e}^{-2zJ_+}J_3 + \frac{1-\mathrm{e}^{-2zJ_+}}{2z}J_- + J_-\frac{1-\mathrm{e}^{-2zJ_+}}{2z} + \mathrm{e}^{-2zJ_+} - 1 \tag{2.16}$$

and the universal R-matrix reads

$$\mathcal{R} = \exp\{-zJ_+ \otimes J_3\}\exp\{zJ_3 \otimes J_+\}. \tag{2.17}$$

A realization of $U_z sl(2,\mathbb{R})$ in terms of the boson algebra $[a_-, a_+] = 1$ reads

$$\begin{aligned}
&J_+ = a_+ \qquad J_3 = \frac{\mathrm{e}^{2za_+} - 1}{z}a_- + \beta\frac{\mathrm{e}^{2za_+} + 1}{2} \\
&J_- = -\frac{\mathrm{e}^{2za_+} - 1}{2z}a_-^2 - \beta\frac{\mathrm{e}^{2za_+} + 1}{2}a_- - z\beta^2\frac{\mathrm{e}^{2za_+} - 1}{8}.
\end{aligned} \tag{2.18}$$

The limit $z \to 0$ of (2.18) gives rise to the GD realization (2.3) for $sl(2,\mathbb{R})$ while the Casimir keeps the same eigenvalue (2.6) along the whole process.

The GD-like quantum formulae (2.18) allow us to directly compute closed expressions for any representation of this quantum algebra (in this respect, the following results can be compared with the more elaborate derivation of such representations given in [17, 18]).

In this way, lower-bounded representations can be obtained from (2.4), by taking into account that

$$\mathrm{e}^{2za_+}|m\rangle = |m\rangle + \sum_{k=1}^{\infty}\frac{(2z)^k}{k!}\sqrt{\frac{(m+k)!}{m!}}|m+k\rangle \tag{2.19}$$

so that the action of (2.18) on the states $\{|m\rangle\}_{m=0}^{\infty}$ for any β is obtained:

$$
\begin{aligned}
J_+|m\rangle &= \sqrt{m+1}|m+1\rangle \\
J_3|m\rangle &= (2m+\beta)|m\rangle + \sum_{k=1}^{\infty}\frac{(2z)^k}{k!}\sqrt{\frac{(m+k)!}{m!}}\left(\frac{2m}{k+1}+\frac{\beta}{2}\right)|m+k\rangle \\
J_-|m\rangle &= -\sqrt{m}(m-1+\beta)|m-1\rangle - \sum_{k=1}^{\infty}\frac{(2z)^k}{k!}\sqrt{\frac{(m+k)!}{m!}} \\
&\qquad\times\left\{\frac{m}{\sqrt{m+k}}\left(\frac{m-1}{k+1}+\frac{\beta}{2}\right)|m-1+k\rangle + \frac{z\beta^2}{8}|m+k\rangle\right\}.
\end{aligned}
\tag{2.20}
$$

For $\beta \equiv \beta_- \in \mathbb{Z}^-$ this action directly provides, by means of a quotient space, the finite-dimensional representations of dimension $|\beta_- - 1|$ in much the same way as for the classical counterpart. Therefore, we will denote $|\beta_- - 1| = 2j_z + 1 \in \mathbb{Z}^+$ being $j_z = 0, \frac{1}{2}, 1 \ldots$. Indeed it is clear that (2.20) contains power series in z whose first terms coincides with the non-deformed analogue shown in (2.5). As the simplest example, we write down the two-dimensional matrix representation with $\beta_- = -1$, $\mathcal{C}_z = \frac{3}{2}$ and $j_z = \frac{1}{2}$ (compare with (2.8)):

$$
J_+ = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \qquad J_- = \begin{pmatrix} \cdot & 1 \\ -\frac{1}{4}z^2 & z \end{pmatrix} \qquad J_3 = \begin{pmatrix} -1 & \cdot \\ -z & 1 \end{pmatrix}. \tag{2.21}
$$

The finite-dimensional representations provide explicit solutions of the quantum YBE. The computations are considerably simplified due to the factorized form of the R-matrix (2.17); the one corresponding to the above example reads

$$
\mathcal{R} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ -z & 1 & \cdot & \cdot \\ z & \cdot & 1 & \cdot \\ z^2 & -z & z & 1 \end{pmatrix}. \tag{2.22}
$$

Upper-bounded representations on polynomial spaces $\{x^n\}_{n=0}^{+\infty}$ are supplied by means of the operator

$$
D_z \equiv (e^{2z\partial_x} - 1)/2z. \tag{2.23}
$$

Note that the action of D_z is just that of a discrete derivative:

$$
D_z\phi(x) = (\phi(x+2z) - \phi(x))/2z.
$$

Now it is easy to check that (2.18) gives rise to the following differential–difference realization of $U_z sl(2,\mathbb{R})$:

$$
\begin{aligned}
J_+ &= \partial_x \qquad J_3 = -2D_z x + z\beta D_z + \beta \\
J_- &= -D_z x^2 + z\beta D_z x - \frac{z^2\beta^2}{4} D_z + \beta x.
\end{aligned}
\tag{2.24}
$$

From these expressions, we see that non-standard quantum deformations are related to a difference calculus quite different to that of standard $U_q sl(2,\mathbb{R})$. In analogy to the classical case, the finite-dimensional representations originated from (2.24) for $\beta_+ - 2 \in \mathbb{Z}^+$ ($\beta_+ - 1 \equiv 2j_z + 1$) are also supported by $\langle 1, x, \ldots, x^{\beta_+ - 2}\rangle$. These representations, simply denoted by j_z, will be applied in some examples in the next section.

3. Tensor product representations and decomposition rules

Given a pair of representations for the $U_z sl(2,\mathbb{R})$ algebra acting on the vector spaces $\mathcal{H}_1$, $\mathcal{H}_2$ the coproduct (2.12) originates a new representation in the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Although the initial representations may be irreducible the final coproduct representation will be in general reducible. We shall see that, for some particular finite-dimensional cases worked out below, the coproduct representation is completely reduced into irreducible components following the same well known rules valid for the classical $sl(2,\mathbb{R})$ integer representations. Using the conventional notation [26], this decomposition can be written as

$$j_z \otimes j_z' = |j_z + j_z'| \oplus |j_z + j_z' - 1| \oplus \cdots \oplus |j_z - j_z'| \tag{3.1}$$

where j_z and j_z' are positive half-integers corresponding to the quantum representations of dimension $2j_z + 1$ or $2j_z' + 1$, respectively. However, the vector basis of the irreducible support subspaces expressed in terms of the original basis (i.e. the Clebsch–Gordan coefficients) become quite different to those of the $sl(2,\mathbb{R})$ Lie algebra due to extra terms containing powers of the deformation parameter z.

We shall examine these features in detail for two simple examples by using the differential realizations given in (2.24) that are particularly easy to handle for computations.

3.1. $\frac{1}{2} \otimes \frac{1}{2}$ *representations*

The representation $j = \frac{1}{2}$ for the Lie algebra $sl(2,\mathbb{R})$ is realized in the polynomial vector space spanned by $\{|-1\rangle = 1, |-2\rangle = x\}$. For $j = 1$ the basis is chosen in the form $\{|-1\rangle = 1, |-2\rangle = x, |-3\rangle = x^2\}$; we shall use the variable y for the second space in the tensor product. The coproduct representation in this case is defined by $\Delta^{\mathrm{clas}}(X) = 1 \otimes U_{1/2}(X) + U_{1/2}(X) \otimes 1$, where X holds for any of the algebra generators and $U_{1/2}$ is for the $j = \frac{1}{2}$-spin representation. The decomposition $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ has support spaces whose basis are

$$\begin{aligned} &\mathcal{H}_1^{\mathrm{clas}} = \langle \boldsymbol{E}_{-1}^{\mathrm{clas}} \equiv 1, \boldsymbol{E}_{-2}^{\mathrm{clas}} \equiv \tfrac{1}{2}(x+y), \boldsymbol{E}_{-3}^{\mathrm{clas}} \equiv xy \rangle \\ &\mathcal{H}_0^{\mathrm{clas}} = \langle \boldsymbol{U}_{-1}^{\mathrm{clas}} \equiv \tfrac{1}{2}(x-y) \rangle. \end{aligned} \tag{3.2}$$

Obviously, the triplet generating $\mathcal{H}_1^{\mathrm{clas}}$ is symmetric under the permutation map $\sigma(a \otimes b) = b \otimes a$, while the singlet underlying $\mathcal{H}_0^{\mathrm{clas}}$ is antisymmetric.

Now, for the non-standard $U_z sl(2,\mathbb{R})$ the coproduct representation is defined according to (2.12), and we consider the value $\beta_+ = 3$ that corresponds to the case $j_z = \frac{1}{2}$ (2.21). The reduction $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ also remains correct. Here, the representation $j_z = 1$ is obtained with $\beta_+ = 4$, while 0 is of course for the trivial representation. Explictly, the invariant vector subspaces $\mathcal{H}_1 = \langle \boldsymbol{E}_{-1}, \boldsymbol{E}_{-2}, \boldsymbol{E}_{-3} \rangle$ and $\mathcal{H}_0 = \langle \boldsymbol{U}_{-1} \rangle$ are as follows in terms of the basis (3.2):

$$\begin{aligned} &\boldsymbol{E}_{-1} = \boldsymbol{E}_{-1}^{\mathrm{clas}} \\ &\boldsymbol{E}_{-2} = \boldsymbol{E}_{-2}^{\mathrm{clas}} \\ &\boldsymbol{E}_{-3} = \boldsymbol{E}_{-3}^{\mathrm{clas}} + \frac{3z^2}{4}\boldsymbol{E}_{-1}^{\mathrm{clas}} + z\boldsymbol{U}_{-1}^{\mathrm{clas}} \\ &\boldsymbol{U}_{-1} = \boldsymbol{U}_{-1}^{\mathrm{clas}} + \frac{z}{2}\boldsymbol{E}_{-1}^{\mathrm{clas}}. \end{aligned} \tag{3.3}$$

Note that symmetry in the basis of $\mathcal{H}_1$ and antisymmetry in $\mathcal{H}_0$ do not hold unless we assume that the permutation map σ transforms the deformation parameter z into $-z$. On the other hand, it can easily be proven that, for all z, the transformation (3.3) is always well defined. Therefore, roots of unity seem to be not privileged for the non-standard deformation.

3.2. $1\otimes\frac{1}{2}$ representations

First we shall supply the basis of $\mathcal{H}_{1\otimes 1/2}=\mathcal{H}^{\rm clas}_{3/2}\oplus\mathcal{H}^{\rm clas}_{1/2}$ for the reduction $1\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}$ in the Lie algebra context. For the $j=\frac{3}{2}$ representation we use the basis $\{|-1\rangle=1, |-2\rangle=x, |-3\rangle=x^2, |-4\rangle=x^3\}$, and the invariant subspaces in the classical tensor product are spanned by

$$\begin{aligned}&\mathcal{H}^{\rm clas}_{3/2}=\langle \boldsymbol{E}^{\rm clas}_{-1}\equiv 1, \boldsymbol{E}^{\rm clas}_{-2}\equiv \tfrac{1}{3}(y+2x), \boldsymbol{E}^{\rm clas}_{-3}\equiv \tfrac{1}{3}(2xy+x^2), \boldsymbol{E}^{\rm clas}_{-4}\equiv x^2y\rangle\\ &\mathcal{H}^{\rm clas}_{1/2}=\langle \boldsymbol{U}^{\rm clas}_{-1}\equiv \tfrac{1}{2}(y-x), \boldsymbol{U}^{\rm clas}_{-2}\equiv \tfrac{1}{2}xy-x^2\rangle.\end{aligned}\tag{3.4}$$

With respect to the deformed quantum algebra $U_z sl(2,\mathbb{R})$ it can be checked directly that its coproduct leads to the same direct sum reduction $1\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}$ on the same polynomial vector space $\mathcal{H}_{1\otimes 1/2}$ but with new invariant subspaces $\mathcal{H}_{3/2}=\langle\boldsymbol{E}_{-1},\boldsymbol{E}_{-2},\boldsymbol{E}_{-3},\mathbf{E}_{-4}\rangle$ and $\mathcal{H}_{1/2}=\langle\boldsymbol{U}_{-1},\boldsymbol{U}_{-2}\rangle$ given by the following deformed change of basis:

$$\begin{aligned}&\boldsymbol{E}_{-1}=\boldsymbol{E}^{\rm clas}_{-1}\\ &\boldsymbol{E}_{-2}=\boldsymbol{E}^{\rm clas}_{-2}\\ &\boldsymbol{E}_{-3}=\boldsymbol{E}^{\rm clas}_{-3}+\frac{3z^2}{4}\boldsymbol{E}^{\rm clas}_{-1}-\frac{2z}{3}\boldsymbol{U}^{\rm clas}_{-1}\\ &\boldsymbol{E}_{-4}=\boldsymbol{E}^{\rm clas}_{-4}+\frac{9z^2}{4}\boldsymbol{E}^{\rm clas}_{-2}-\frac{9z^3}{4}\boldsymbol{E}^{\rm clas}_{-1}-2z\boldsymbol{U}^{\rm clas}_{-2}-\frac{z^2}{3}\boldsymbol{E}^{\rm clas}_{-1}\\ &\boldsymbol{U}_{-1}=\boldsymbol{U}^{\rm clas}_{-1}-\frac{z}{2}\boldsymbol{E}^{\rm clas}_{-2}\\ &\boldsymbol{U}_{-2}=\boldsymbol{U}^{\rm clas}_{-2}-\frac{3z}{8}\boldsymbol{E}^{\rm clas}_{-3}+\frac{3z^2}{8}\boldsymbol{E}^{\rm clas}_{-2}.\end{aligned}\tag{3.5}$$

As expected, the limit $z\to 0$ provides the classical partners of the reduction process.

At this point it is worth mentioning that representations of the standard deformation of $sl(2,\mathbb{R})$ are strongly different from their non-standard counterparts. On one hand, such standard representations can be essentially constructed by substituting some matrix elements of the classical matrices by the corresponding q-numbers [27] and, consequently, the same holds for the Clebsch–Gordan coefficients. This straightforward method is no longer valid for the non-standard case, where q-numbers do not work and, moreover, some new non-vanishing Clebsch–Gordan coefficients have to be added with respect to the classical theory. Since q-numbers are directly related to the peculiarities at roots of unity, the loss of such properties in the non-standard case seems quite natural.

4. Two-boson $U_z sl(2,\mathbb{R})$ representations and their contraction to Poincaré

The one-boson representations of $sl(2,\mathbb{R})$ are closely linked with its geometrical interpretation as a one-dimensional conformal algebra. In contrast, a description in terms of two-boson algebras is physically related to its role as a $(1+1)$-dimensional kinematical algebra. This fact allows us to perform a contraction in order to reach the quantum $(1+1)$ Poincaré algebra representations; such a process cannot be applied onto the one-boson representations described in section 2.

We consider two independent boson algebras

$$[a_-,a_+]=1\qquad [b_-,b_+]=1.\tag{4.1}$$

A two-boson GD representation of $U_z sl(2,\mathbb{R})$ takes the form

$$J_+ = a_+ \qquad J_3 = \frac{e^{2za_+}-1}{z}a_- - 2b_+b_- \\ J_- = -\frac{e^{2za_+}-1}{2z}a_-^2 + 2b_+b_-a_- + \alpha b_+ + 2z(b_+b_- + b_+^2b_-^2). \tag{4.2}$$

The limit $z \to 0$ gives rise to the classical two-boson GD realization

$$J_+ = a_+ \qquad J_3 = 2a_+a_- - 2b_+b_- \\ J_- = -a_+a_-^2 + 2b_+b_-a_- + \alpha b_+ \tag{4.3}$$

which should be compared with the more common Jordan–Schwinger one [28].

Algebraically, $U_z sl(2,\mathbb{R})$ can be contracted to a non-standard $(1+1)$ Poincaré algebra, $U_z\mathcal{P}(1+1)$, by defining the generators as

$$P_+ = \varepsilon J_+ \qquad P_- = \varepsilon J_- \qquad K = \tfrac{1}{2}J_3 \tag{4.4}$$

and by transforming the deformation parameter in the form

$$z \to \varepsilon^{-1}z \tag{4.5}$$

so that we obtain the following $(1+1)$ Poincaré deformed Hopf algebra:

$$\Delta(P_+) = 1\otimes P_+ + P_+\otimes 1 \\ \Delta(K) = 1\otimes K + K\otimes e^{2zP_+} \\ \Delta(P_-) = 1\otimes P_- + P_-\otimes e^{2zP_+} \tag{4.6}$$

$$\epsilon(X) = 0 \qquad \text{for } X\in\{K, P_+, P_-\} \tag{4.7}$$

$$\gamma(P_+) = -P_+ \qquad \gamma(K) = -Ke^{-2zP_+} \qquad \gamma(P_-) = -P_-e^{-2zP_+} \tag{4.8}$$

$$[K, P_+] = \frac{e^{2zP_+}-1}{2z} \qquad [K, P_-] = -P_- \qquad [P_+, P_-] = 0. \tag{4.9}$$

The quantum Casimir is found by contracting (2.16) as the $\lim_{\varepsilon\to 0}(\varepsilon^2\mathcal{C}_z)$ and the R-matrix comes directly from the contraction of (2.17):

$$C_z = \frac{1-e^{-2zP_+}}{z}P_- \tag{4.10}$$

$$R = \exp\{-2zP_+\otimes K\}\exp\{2zK\otimes P_+\}. \tag{4.11}$$

The corresponding classical r-matrix is $r = 2zK\wedge P_+$. These results were obtained in [20] by following a T-matrix approach.

In order to contract representation (4.2) we consider (4.4), (4.5) together with [29]:

$$a_- \to \varepsilon^{-1}a_- \qquad a_+ \to \varepsilon a_+ \qquad b_- \to b_- \qquad b_+ \to b_+ \qquad \alpha \to \varepsilon\alpha. \tag{4.12}$$

After the limit $\varepsilon \to 0$ the contracted two-boson representation becomes

$$P_+ = a_+ \qquad K = \frac{e^{2za_+}-1}{2z}a_- - b_+b_- \qquad P_- = \alpha b_+. \tag{4.13}$$

The kinematical interpretation underlying the quantum representations (4.2) and (4.13) can be enlightened by considering the corresponding differential realizations provided by

$$a_- = -x^+ \qquad a_+ = \frac{\partial}{\partial x^+} \equiv \partial_+ \qquad b_- = -x^- \qquad b_+ = \frac{\partial}{\partial x^-} \equiv \partial_- \tag{4.14}$$

where, $x^+ = t + x$ and $x^- = t - x$ can be identified, in the classical case, as light-cone coordinates.

5. Extended $U_z sl(2,\mathbb{R})$ and its contraction to the oscillator algebra

The infinite-dimensional representations for the non-standard quantum oscillator algebra [21] can be deduced by performing a contraction on a (non-standard) quantum deformation of the pseudo-extended $sl(2,\mathbb{R})$ Lie algebra. In this section we develop such a process at both classical and quantum levels.

5.1. Classical level

It is well known that the following Lie algebra

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = J_3 - I \qquad [I, \cdot] = 0 \tag{5.1}$$

is a trivial central extension of $sl(2,\mathbb{R})$. The generator I commutes with the original $sl(2,\mathbb{R})$ generators, and 'trivial' means that the algebra (5.1), hereafter denoted by $\overline{sl}(2,\mathbb{R})$, is isomorphic to the direct sum $sl(2,\mathbb{R}) \oplus \mathbb{R}$ through the change of basis $J_3' = J_3 - I$. The second-order Casimir for $\overline{sl}(2,\mathbb{R})$ is:

$$\mathcal{C} = \tfrac{1}{2}J_3^2 - J_3 I + J_+J_- + J_-J_+. \tag{5.2}$$

The interest of the algebra $\overline{sl}(2,\mathbb{R})$ comes from the fact that it leads, through a careful contraction, to a non-trivial extension of the $(1+1)$ Poincaré algebra corresponding to a constant non-zero background field [30]; this contracted extended algebra is isomorphic to the oscillator h_4 Lie algebra.

The one-boson realization for (5.1) is

$$J_+ = a_+ \qquad J_3 = 2a_+a_- + \beta 1 \qquad J_- = -a_+a_-^2 - \beta a_- + \delta a_- \qquad I = \delta 1 \tag{5.3}$$

where δ and β are free parameters related with the eigenvalue of the Casimir by $\mathcal{C} = \beta(\beta/2 - 1) + \delta(1-\beta)$. The contraction we are interested in is defined by

$$A_+ = \varepsilon J_+ \qquad A_- = \varepsilon J_- \qquad N = J_3/2 \qquad M = \varepsilon^2 I \tag{5.4}$$

so that in the limit $\varepsilon \to 0$ we get the oscillator h_4 Lie algebra,

$$[N, A_+] = A_+ \qquad [N, A_-] = -A_- \qquad [A_-, A_+] = M \qquad [M, \cdot] = 0. \tag{5.5}$$

The corresponding second-order Casimir is obtained as $\lim_{\varepsilon\to 0}(-\varepsilon^2\mathcal{C})$:

$$C = 2NM - A_+A_- - A_-A_+. \tag{5.6}$$

The additional replacements

$$a_- \to \varepsilon^{-1}a_- \qquad a_+ \to \varepsilon a_+ \qquad \beta \to \beta/2 \qquad \delta \to \varepsilon^2\delta \tag{5.7}$$

provide the one-boson h_4 realization:

$$N = a_+a_- + \beta \qquad A_+ = a_+ \qquad A_- = \delta a_- \qquad M = \delta 1. \tag{5.8}$$

Hence the Casimir eigenvalue is $C = \delta(2\beta - 1)$. Note that (5.8) makes the difference clearer between considering a pure boson algebra and the full harmonic oscillator algebra h_4.

5.2. Quantum level: The non-standard oscillator

In the quantum context we closely follow the classical approach by defining an appropriate quantum deformation of $\overline{sl}(2,\mathbb{R})$ that must include the new generator I. Such a new quantum algebra is given by

$$\begin{aligned}
&\Delta(J_+) = 1\otimes J_+ + J_+\otimes 1\\
&\Delta(J_3) = 1\otimes J_3 + J_3\otimes \mathrm{e}^{2zJ_+}\\
&\Delta(J_-) = 1\otimes J_- + J_-\otimes \mathrm{e}^{2zJ_+} + zJ_3\otimes I\mathrm{e}^{2zJ_+}\\
&\Delta(I) = 1\otimes I + I\otimes 1
\end{aligned}\tag{5.9}$$

$$\epsilon(X) = 0 \qquad \text{for } X\in\{J_3, J_+, J_-, I\}\tag{5.10}$$

$$\begin{aligned}
&\gamma(J_+) = -J_+ \qquad \gamma(I) = -I \qquad \gamma(J_3) = -J_3\mathrm{e}^{-2zJ_+}\\
&\gamma(J_-) = -J_-\mathrm{e}^{-2zJ_+} + zJ_3 I\mathrm{e}^{-2zJ_+}
\end{aligned}\tag{5.11}$$

$$\begin{aligned}
&[J_3, J_+] = \frac{\mathrm{e}^{2zJ_+}-1}{z} \qquad [J_3, J_-] = -2J_- + zJ_3^2\\
&[J_+, J_-] = J_3 - I\mathrm{e}^{2zJ_+} \qquad [I, \cdot] = 0.
\end{aligned}\tag{5.12}$$

The (coboundary) Lie bialgebra underlying this Hopf algebra is again generated by $r = zJ_3\wedge J_+$. Note that the new generator I remains central and primitive; there is another quantum Casimir given by

$$\mathcal{C}_z = \frac{1}{2}J_3\mathrm{e}^{-2zJ_+}J_3 - J_3 I + \frac{1-\mathrm{e}^{-2zJ_+}}{2z}J_- + J_-\frac{1-\mathrm{e}^{-2zJ_+}}{2z} + \mathrm{e}^{-2zJ_+} - 1.\tag{5.13}$$

The Hopf subalgebra generated by J_3 and J_+ is the same as in the non-extended case, therefore, the universal R-matrix (2.17) is obviously a solution of the quantum YBE for $U_z\overline{sl}(2,\mathbb{R})$. Furthermore, cumbersome computations show that this R-matrix verifies $\mathcal{R}\Delta(J_-)\mathcal{R}^{-1} = \sigma\circ\Delta(J_-)$ in $U_z\overline{sl}(2,\mathbb{R})$ (the proof for I is trivial), so, we conclude that it is also a universal R-matrix for the whole $U_z\overline{sl}(2,\mathbb{R})$. At this point it is worth mentioning that another quantum deformation of the extended $\overline{sl}(2,\mathbb{R})$ has recently been proposed in [31] leading to a deformed oscillator algebra with classical commutation relations (however, this deformation does not preserve the aforementioned subalgebra).

The one-boson realization of $U_z\overline{sl}(2,\mathbb{R})$ turns out to be a slight modification of the non-extended case (2.18). Besides the new generator $I = \delta 1$, the only generator that must be changed is J_-:

$$J_- = -\frac{\mathrm{e}^{2za_+}-1}{2z}a_-^2 - \beta\frac{\mathrm{e}^{2za_+}+1}{2}a_- - z\beta^2\frac{\mathrm{e}^{2za_+}-1}{8} + \delta\mathrm{e}^{2za_+}a_- + \frac{\beta z}{2}\delta\mathrm{e}^{2za_+}.\tag{5.14}$$

Now we proceed to carry out the contraction from $U_z\overline{sl}(2,\mathbb{R})$ to the non-standard quantum oscillator algebra [21], denoted $U_z h_4$. At the Hopf algebra level, we consider the new generators defined by (5.4) and also the transformation of the deformation parameter z (4.5). Thus, when $\varepsilon\to 0$ we arrive at the Hopf structure of $U_z h_4$ given by

$$\begin{aligned}
&\Delta(A_+) = 1\otimes A_+ + A_+\otimes 1 \qquad \Delta(M) = 1\otimes M + M\otimes 1\\
&\Delta(N) = 1\otimes N + N\otimes \mathrm{e}^{2zA_+}\\
&\Delta(A_-) = 1\otimes A_- + A_-\otimes \mathrm{e}^{2zA_+} + 2zN\otimes M\mathrm{e}^{2zA_+}
\end{aligned}\tag{5.15}$$

$$\epsilon(X) = 0 \qquad X\in\{N, A_+, A_-, M\}\tag{5.16}$$

$$\begin{aligned}
&\gamma(A_+) = -A_+ \qquad \gamma(M) = -M\\
&\gamma(N) = -N\mathrm{e}^{-2zA_+} \qquad \gamma(A_-) = -A_-\mathrm{e}^{-2zA_+} + 2zNM\mathrm{e}^{-2zA_+}
\end{aligned}\tag{5.17}$$

satisfying the commutators

$$
\begin{aligned}
&[N, A_+] = \frac{e^{2zA_+} - 1}{2z} \\
&[N, A_-] = -A_- \\
&[A_-, A_+] = M e^{2zA_+} \\
&[M, \cdot] = 0
\end{aligned}
\tag{5.18}
$$

where the classical r-matrix is $r = 2zN \wedge A_+$. Besides the generator M there is another central operator which is directly obtained from (5.13) by means of the $\lim_{\varepsilon \to 0}(-\varepsilon^2 \mathcal{C}_z)$ giving rise to the expression

$$
C_z = 2NM + \frac{e^{-2zA_+} - 1}{2z} A_- + A_- \frac{e^{-2zA_+} - 1}{2z}. \tag{5.19}
$$

Likewise, the corresponding universal R-matrix is found by contracting (2.17):

$$
R = \exp\{-2zA_+ \otimes N\} \exp\{2zN \otimes A_+\}. \tag{5.20}
$$

The boson representation of $U_z h_4$ can be found using the same routine taking into account (5.4), (4.5) plus the extra replacements (5.7). Thus we have

$$
\begin{aligned}
&A_+ = a_+ \qquad M = \delta 1 \\
&A_- = \delta e^{2za_+} a_- + \delta \beta z e^{2za_+} \\
&N = \frac{e^{2za_+} - 1}{2z} a_- + \beta \frac{e^{2za_+} + 1}{2}.
\end{aligned}
\tag{5.21}
$$

Hence, the action on the states $\{|m\rangle\}_{m=0}^{\infty}$ reads

$$
\begin{aligned}
&A_+|m\rangle = \sqrt{m+1}|m+1\rangle \qquad M|m\rangle = \delta|m\rangle \\
&A_-|m\rangle = \delta\sqrt{m}|m-1\rangle + \delta \sum_{k=0}^{\infty} \frac{(2z)^{k+1}}{k!} \sqrt{\frac{(m+k)!}{m!}} \left(\frac{m}{k+1} + \frac{\beta}{2} \right) |m+k\rangle \\
&N|m\rangle = (m+\beta)|m\rangle + \sum_{k=1}^{\infty} \frac{(2z)^k}{k!} \sqrt{\frac{(m+k)!}{m!}} \left(\frac{m}{k+1} + \frac{\beta}{2} \right) |m+k\rangle.
\end{aligned}
\tag{5.22}
$$

The explicit infinite-dimensional representations in the monomial basis $\{x^n\}$ as well as a differential–difference realization in terms of operator (2.23) can now be readily obtained.

6. Concluding remarks

In this paper we have given a unified treatment for a class of non-standard quantum algebras related to $U_z sl(2, \mathbb{R})$: its deformed extension $U_z \overline{sl}(2, \mathbb{R})$, the Poincaré algebra $U_z \mathcal{P}(1+1)$, and the non-standard oscillator $U_z h_4$. All these algebras share the same Hopf subalgebra (in the $U_z sl(2, \mathbb{R})$ case is generated by $\{J_3, J_+\}$) which leads to a formally identical universal R-matrix for all of them.

At the same time we have computed the representations of these non-standard algebras by means of boson operators. We would like to emphasize that, although the use of either one- or two-boson realizations depends on the algebra considered, the generalizations of the GD map here presented seems to be in general the appropriate non-standard bosonization method, in contrast to the usual Jordan–Schwinger map, that turns out to be more adequate for standard deformations. We have obtained simple closed expressions and applicable differential realizations for the $U_z sl(2, \mathbb{R})$ representations (see also in this respect [17]) which parallel the Lie algebraic classification. For instance, finite-dimensional representations have been shown to be labelled by integers j_z and we have proven how their coproduct representations decompose by following exactly the classical addition of angular momenta [26].

Contraction processes relating the (one- or two-) bosonic representations of all these non-isomorphic quantum algebras have been found. Careful attention was paid to define the most adequate non-standard quantum deformation of the centrally pseudo-extended algebra $\overline{sl}(2, \mathbb{R})$. Indeed, it has some original features with respect to other extensions already defined in the context of quantum deformations (for instance, the way in which the central extension generator enters into the deformed commutation rules). This extension allows for a contraction to $U_z h_4$ that preserves the Hopf subalgebra $\{J_3, J_+\}$.

Finally, we would like to notice that the deformed boson algebra defined by

$$\bar{a}_+ = \frac{\mathrm{e}^{2za_+} - 1}{2z} \qquad \bar{a}_- = a_- + \mu z \qquad [\bar{a}_-, \bar{a}_+] = 1 + 2z\bar{a}_+ \tag{6.1}$$

allows us to obtain quadratic commutation rules for all these quantum algebras, once a suitable change of the basis generators and a precise value of μ have been chosen for each case.

Acknowledgment

This work was partially supported by DGICYT (projects PB94–1115 and PB95–0719) from the Ministerio de Educación y Cultura de España.

References

[1] Caspers W J 1989 *Spin Systems* (Singapore: World Scientific)
[2] Klein A and Marshalek E R 1991 *Rev. Mod. Phys.* **63** 375
[3] Dyson J F 1956 *Phys. Rev.* **102** 1217
[4] Kulish P P and Reshetikhin N Y 1983 *J. Sov. Math.* **23** 2435
[5] Drinfel'd V 1986 Quantum groups *Proc. Int. Cong. of Mathematics* (Berkeley, CA: MRSI)
[6] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[7] Chari V and Pressley A 1995 *A guide to Quantum Groups* (Cambridge: Cambridge University Press)
[8] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[9] Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287
[10] Ohn C 1992 *Lett. Math. Phys.* **25** 85
[11] Ballesteros A, Herranz F J, del Olmo M A, and Santander M 1995 *J. Phys. A: Math. Gen.* **28** 941
[12] Karimipour V 1994 *Lett. Math. Phys.* **30** 87
[13] de Azcárraga J A and Ródenas F 1997 *Quantum Groups and Quantum Spaces* vol 40, ed P Budzyski, W Pusz and S Zakrzewski (Banach) p 1
[14] Shariati A, Aghamohammadi A and Khorrami M 1996 *Mod. Phys. Lett.* **11A** 187
[15] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** L311
[16] Dobrev V K 1996 Representations of the Jordanian quantum algebra $U_h(sl(2))$ *Proc. 10th Int. Conf. Problems of Quantum Field Theory* ed D Shirkov, D Kazakov and A Vladimirov JINR E2-96-369 (Dubna) pp 104 *preprint* ICTP IC/96/14
[17] Abdesselam B, Chakrabarti A and Chakrabarti R 1996 *Mod. Phys. Lett.* A **11** 2883
[18] Abdesselam B, Chakrabarti A and Chakrabarti R 1997 *Int. J. Mod. Phys.* A **12** 2301
[19] Khorrami M, Shariati A, Abolhassani M R and Aghamohammadi A 1995 *Mod. Phys. Lett.* A **10** 873
[20] Ballesteros A, Herranz F J, del Olmo M A, Pereña C M and Santander M 1995 *J. Phys. A: Math. Gen.* **28** 7113
[21] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** 4307
[22] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[23] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581

[24] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys.* **148** 346
[25] Vilenkin N Ya and Klimyk A U 1991 *Representations of Lie Groups and Special Functions* (Dordrecht: Kluwer)
[26] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics (Encyclopedia of Mathematics)* (New York: Addison-Wesley)
[27] Biedenharn L C and Lohe M A 1995 *Quantum Group Symmetry and q-tensor Algebras* (Singapore: World Scientific)
[28] Schwinger J 1952 On angular momentum *US At. Energy Comm.* NYO–3071
[29] de Montigny M 1993 *Nuovo Cimento* B **108** 1171
[30] Negro J and del Olmo M A 1993 *J. Math. Phys.* **34** 1206
[31] Abdesselam B The twisted Heisenberg algebra $U_{h,w}(H(4))$ *J. Math. Phys.* to appear